University of Richmond
Robins School of Business™

§9. Efficient Diversification

FIN 366: Investments

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PORTFOLIO RISK AND DIVERSIFICATION

The sources of uncertainty and risk inherent to investing in a security are classified into two broad categories:

1. **Firm Specific** or **Nonsystematic** or **Diversifiable Risk** is related to the firm itself, such as the possibility of bankruptcy, manager fraud, product failure, or a favorable drop in the price of inputs.

2. **Market-wide** or **Systematic** or **Nondiversifiable Risk** affects the market as a whole and may include pandemics, interest rates, trade tensions, housing crises, or favorable employment conditions.

An investor who purchases shares of just one firm is exposed to both these sources of risk. If an investor purchases shares of different firms to develop a risky portfolio of multiple assets, firm specific risk becomes less relevant, given different securities can move in different ways.

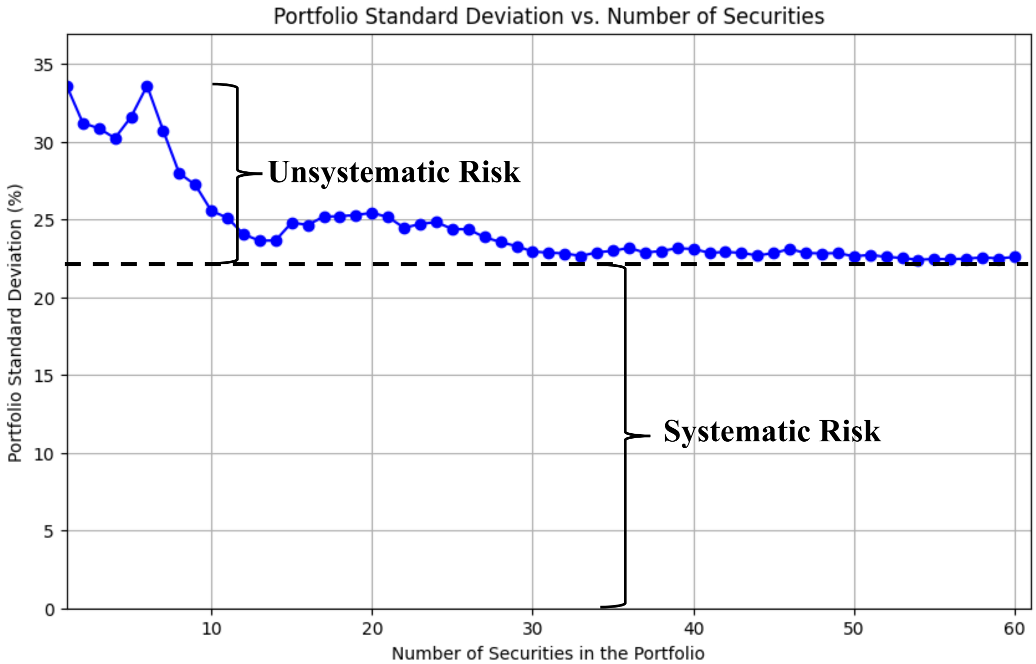
DIVERSIFICATION

Diversification is the spreading of a portfolio over many securities to avoid excessive exposure to any one source of firm-specific risk.

- Diversification *can* eliminate unique, firm-specific, diversifiable, or non-systematic risk.

- Diversification *cannot* eliminate market, systematic, non-diversifiable risk.

Figure 1: Diversifiable and Systematic Risk



This figure shows the standard deviation of various equal-weight portfolios beginning with one stock (on the left), and incrementally adding one stock at a time until there are 60 stocks in the portfolio (on the right). The risk of the portfolio overall is not significantly decreased beyond about 30 securities.

Generate a figure using real-world financial data at *Python Code: Unsystematic and Systematic Risk*, available at josephfarizo.com/fin366.html.

CORRELATION AND COVARIANCE[1]

As we construct diversified portfolios, we take into account the correlation between assets. The **correlation** of two assets measures the strength of the relationship between their relative movements. The **correlation coefficient** $\boldsymbol{\rho}$ (rho) falls between -1 and 1, from perfect negative correlation (always moving apart) to perfect positive correlation (always moving together).

> Adding securities with low or negative correlations improves the diversification benefit. The securities' price movements are less likely to be in sync. Good performance in some securities offsets poor performance from others.

We compute the correlation coefficient as

$$\rho_{1,2} = \frac{Cov(r_1, r_2)}{\sigma_1 \sigma_2}$$

where the **covariance** $Cov(r_1, r_2)$, a measure of *joint variability* of risk between two assets, is calculated as

$$Cov(r_1, r_2) = \sum_{i=1}^{n} p(i)[r_1(i) - E(r_1)][r_2(i) - E(r_2)]$$

The covariance is the sum across all states of the product of each asset's return minus its expected return and the probability of that state occurring.

PRACTICE: Assume we have a two-asset risky portfolio consisting of a stock fund **S** and a bond fund **B**. The returns of the stock and bond fund, respectively, in a severe recession, mild recession, normal growth period, and economic boom are -37% and -9%, -11% and 15%, 14% and 8%, and 30% and -5%. The probabilities of these states, from severe recession to boom are 5%, 25%, 40%, and 30%. Determine the covariance and correlation coefficient for these two funds.

We will use the Excel file *Calculating Covariance and Correlation* at josephfarizo.com/fin366.html for this example (and to provide additional examples).

SOLUTION: As we've done previously with scenario analysis, it helps to summarize what we know in a table.

Scenario	p(s)	r(**S**)	r(**B**)
Severe Recession	0.05	-0.37	-0.09
Mild Recession	0.25	-0.11	0.15
Normal Growth	0.40	0.14	0.08
Economic Boom	0.30	0.30	-0.05

From the covariance formula, we know that we need to compute the expected returns for each of the stock and bond funds and subtract these expected returns from the returns in each state.

$$Cov(r_S, r_B) = \sum_{i=1}^{n} p(i)[r_S(i) - E(r_S)][r_B(i) - E(r_B)]$$

That is, we sum the product of the probabilities and returns in each state or scenario:

Scenario	p(s)	r(**S**)	p(s) × r(**S**)	r(**B**)	p(s) × r(**B**)
Severe Recession					
Mild Recession					
Normal Growth					
Economic Boom					
		$E(r_S)$=		$E(r_B)$=	

Next, we obtain the deviations from the expected values for the returns of the stock and bond fund in each state.

Scenario	p(s)	r(**S**)	p(s) × r(**S**)	Dev. S	r(**B**)	p(s) × r(**B**)	Dev. B
Severe Recession							
Mild Recession							
Normal Growth							
Economic Boom							
		$E(r_S)$=			$E(r_B)$=		

Next, we take the product of the **p(s)**, **Deviation S**, and **Deviation B** columns, then sum to obtain the covariance.

Scenario	p(s)	r(**S**)	p(s) × r(**S**)	Dev. **S**	r(**B**)	p(s) × r(**B**)	Dev. **B**	Product
Severe Recession								
Mild Recession								
Normal Growth								
Economic Boom								
		E(rS)=			E(rB)=		COV =	

INTERPRETATION: The covariance measure provides a measure of the joint variability of the returns of each of these funds. However, we may wish to convert to the easier-to-interpret correlation coefficient:

$$\rho_{1,2} = \frac{Cov(r_1, r_2)}{\sigma_1 \sigma_2} = \frac{Cov(r_S, r_B)}{\sigma_S \sigma_B}$$

This requires that we compute the standard deviations of the stock fund and bond fund using scenario analysis as we've done before. An abbreviated table is presented below, with the probabilities and deviations for each fund from above.

Scenario	p(s)	Dev. **S**	Dev. **B**	(Dev. **S**)^2	(Dev. **B**)^2	p(s) × (Dev. **S**)^2	p(s) × (Dev. **B**)^2
Severe Recession							
Mild Recession							
Normal Growth							
Economic Boom							
					Variance =		
					St. Dev. =		

And now, we compute our correlation as:

$$\rho_{1,2} = \frac{Cov(r_1, r_2)}{\sigma_1 \sigma_2} = \frac{Cov(r_S, r_B)}{\sigma_S \sigma_B} = \frac{\qquad}{\quad \times \quad} =$$

ASSET ALLOCATION WITH TWO RISKY ASSETS

Assume we have a portfolio of two assets 1 and 2, such as a stock fund and a bond fund. The return of a portfolio r_p that consists of these two assets is a weighted average of the returns r:

$$r_p = w_1 r_1 + w_2 r_2$$

Where w_1 and w_2 are the weights of assets 1 and 2. Similarly, the *expected return* of a portfolio of these two assets is the *expected* return of the two funds:

$$E(r_p) = w_1 E(r_1) + w_2 E(r_2)$$

Quantifying risk, however, requires that we consider not only their respective weights, but also the **correlation** between the returns of the two assets. That is, the *variance* of a portfolio is:

$$\sigma_p^2 = (w_1 \sigma_1)^2 + (w_2 \sigma_2)^2 + 2 w_1 w_2 Cov(r_1, r_2)$$

And the standard deviation of the portfolio is:

$$\sigma_p = \sqrt{(w_1 \sigma_1)^2 + (w_2 \sigma_2)^2 + 2 w_1 w_2 Cov(r_1, r_2)}$$

Given the relationship between the correlation and the covariance, we can similarly substitute in the correlation coefficient for the covariance:

$$\rho_{1,2} = \frac{Cov(r_1, r_2)}{\sigma_1 \sigma_2} \Rightarrow Cov(r_1, r_2) = \rho_{1,2}\sigma_1\sigma_2$$

$$\sigma_p = \sqrt{(w_1\sigma_1)^2 + (w_2\sigma_2)^2 + 2w_1w_2Cov(r_1, r_2)}$$

$$\sigma_p = \sqrt{(w_1\sigma_1)^2 + (w_2\sigma_2)^2 + 2w_1w_2\rho_{1,2}\sigma_1\sigma_2}$$

Rewriting,

$$\sigma_p = \sqrt{(w_1\sigma_1)^2 + (w_2\sigma_2)^2 + 2w_1\sigma_1w_2\sigma_2\rho_{1,2}}$$

Crucially, it is important to observe that the risk of this two-risky-asset portfolio includes:

1. weights of the securities in the portfolio (*w*)
2. their individual levels of risk (σ)
3. the correlation between the returns of the two assets (ρ)

The third point in particular has a powerful implication for diversification. To illustrate, consider the following example of a stock fund and bond fund portfolio.

EXAMPLE: Suppose an investor estimates the following: the expected return of a bond fund is 5% with a standard deviation of 8%. The expected return of a stock fund is 10%, with a standard deviation of 19%. The correlation between the stock fund and bond fund is 0.2. Given these estimates, the investor would like to create a portfolio consisting of 85% in the bond fund and 15% in the stock fund.

We have the following inputs:

Bond fund: $E(r_B) = E(r_1) = 5\%$ $\sigma_B = \sigma_1 = 8\%$ $w_B = w_1 = 85\%$

Stock fund: $E(r_s) = E(r_2) = 10\%$ $\sigma_S = \sigma_2 = 19\%$ $w_S = w_2 = 15\%$

Correlation: $\rho_{1,2} = \rho_{B,S} = 0.2$

The expected return for the risky portfolio is:

$$E(r_p) = w_1 E(r_1) + w_2 E(r_2) = (0.85)(0.05) + (0.15)(0.10) = 5.75\%$$

The standard deviation for the risky portfolio is:

$$\sigma_p = \sqrt{(w_1\sigma_1)^2 + (w_2\sigma_2)^2 + 2w_1\sigma_1 w_2\sigma_2\rho_{1,2}}$$

$$= \sqrt{(0.85 \times 0.08)^2 + (0.15 \times 0.19)^2 + 2(0.85 \times 0.08)(0.15 \times 0.19)(0.2)}$$

$$= 7.88\% \quad (\text{which is } < 8\% \text{ and } < 19\%)$$

INTERPRETATION: For this case, **the standard deviation of the risky portfolio is lower than the standard deviations of either of the individual securities.** Because of the correlation between these two assets, it is possible to combine the two risky assets and lower the overall risk *below that of either of them individually.*

Diversification entails that it is possible for an investor to combine risky assets in such a way that the overall risk of the portfolio can be lower than any one asset's individual risk.

THE INVESTMENT OPPORTUNITY SET

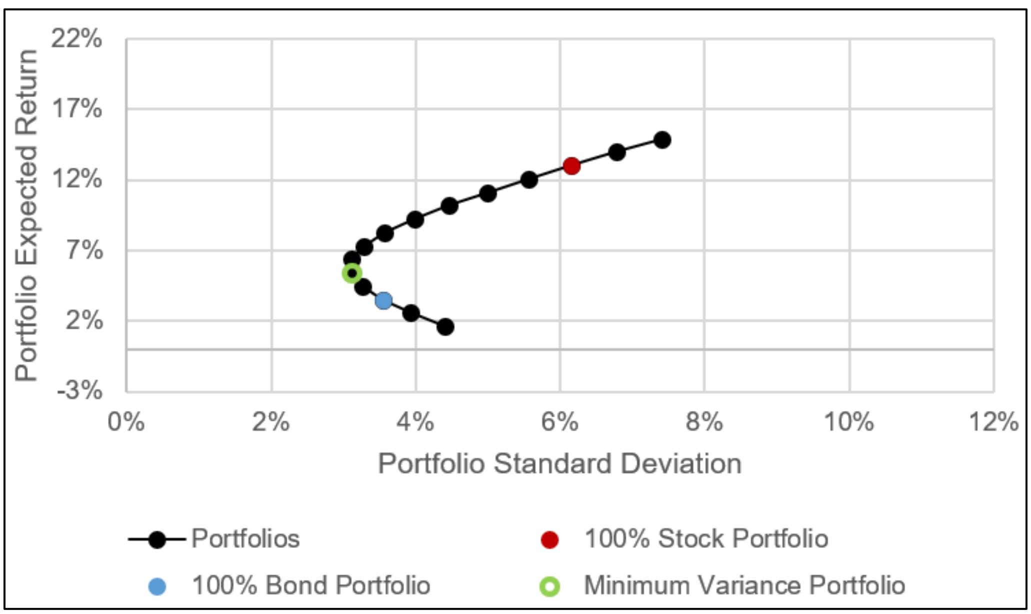
We have seen how to compute the returns, expected returns, and risk for a two-asset portfolio, in this case a stock and bond fund. By varying the weights of our two assets, we can create different portfolios.

For example, a portfolio consisting of 25% in the Vanguard VBTLX bond fund and 75% in the Vanguard VTSAX stock fund has different characteristics in terms of its return, expected return, and risk than a portfolio consisting of 75% in the *same* Vanguard VBTLX bond fund and 25% in the *same* Vanguard VTSAX stock fund.

> Changing the *weights* changes the *portfolio*, even if the risky assets are the same.

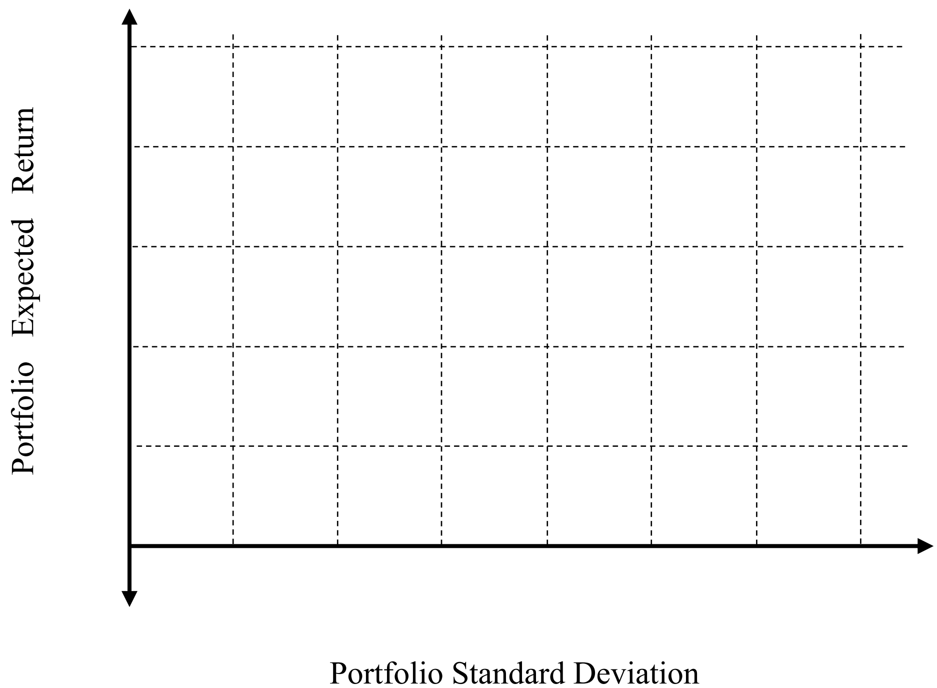
The **Investment Opportunity Set (IOS)** is the collection of possible expected return and risk characteristics that are achievable by altering the weights to the risky assets. It is drawn on a coordinate plane with the *y-axis* as the portfolio expected return and the *x-axis* the portfolio risk.

Figure 2: IOS of a Stock Fund and a Bond Fund



> We will illustrate the investment opportunity set for a randomly generated stock fund and bond fund using the Excel file *Investment Opportunity Set* available at josephfarizo.com/fin366.html. Use the blank axes below to take notes accordingly.

DRAWING THE INVESTMENT OPPORTUNITY SET



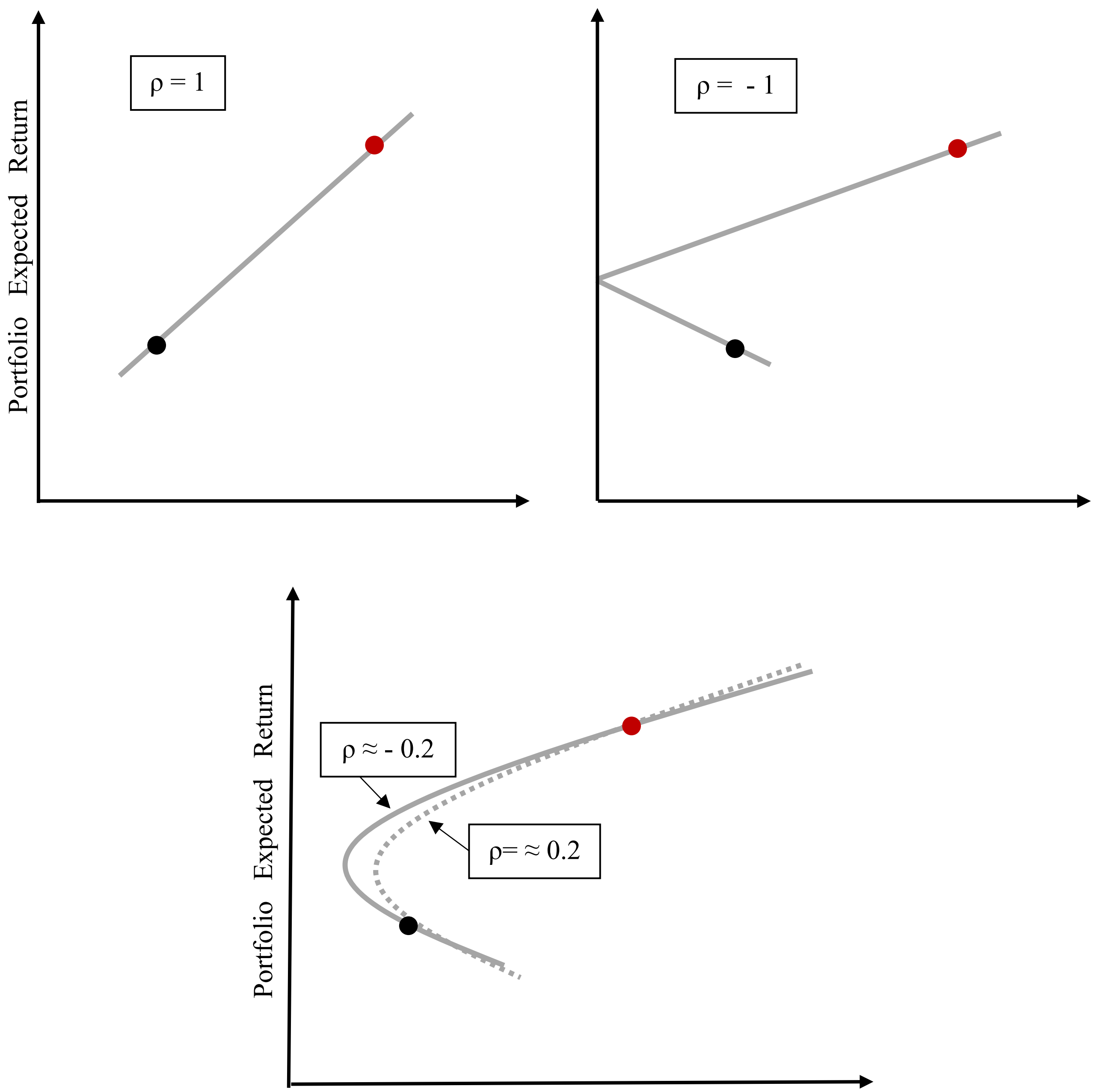
- Each point on the IOS represents a portfolio. By varying the weights, we can plot infinitely many portfolios represented by additional points and the curve connecting these points.

- The *leftmost* point on the IOS represents the **minimum variance portfolio**, or the portfolio with the lowest level of risk.[2]

THE SHAPE OF THE IOS

The correlation between the two assets in the portfolio defines the curvature of the investment opportunity set. With *perfect* negative correlation, there is theoretically a portfolio that can be constructed with no risk and positive return.

Figure 3: Correlations and Shapes of the IOS



The Mean Variance Criterion

We can evaluate the many portfolios on the IOS given their risk and return profiles. There are two components to consider: reward *and* risk. Therefore, we need to consider both to determine the best portfolio, by the **Mean Variance Criterion**:

Portfolio 1 dominates Portfolio 2 if and only if:

$$\mathbf{E(R_1) \geq E(R_2)} \textit{ and } \boldsymbol{\sigma_1 \leq \sigma_2}$$

Or, both Portfolio 1's expected return is greater and risk is lower than Portfolio 2.

Modern Portfolio Theory

This framework of maximizing return for a given level of risk (or **mean-variance optimization**) by considering expected returns, standard deviations, and correlations is known as **Modern Portfolio Theory** (**MPT**). **Harry Markowitz** developed MPT in the 1950s, later winning the Nobel Memorial Prize in Economic Sciences for his work.

CRITICAL THINKING QUESTIONS

1. What type of risk can be effectively eliminated when we invest? What type of risk is always present? Provide examples of each type of risk.
2. Why is the possibility of *good* news considered risk? How is risk defined?
3. Diversification entails spreading our portfolio over many risky assets. Why diversify in the first place? Shouldn't we just hold the asset with the highest possible Sharpe ratio?
4. Your friend tells you "I think my financial advisor is trying to mislead me. He keeps telling me that I can lower my overall risk if I add some stocks to my bond portfolio. But stocks are riskier than bonds…there's just no way that this is true!" What do you tell your friend?
5. The expected return of a portfolio is a weighted average of the component assets. Why, then, is the risk of a portfolio *not* just a weighted average of the individual asset standard deviations?
6. If you want to better diversify, should you buy portfolios of securities that are negatively correlated or highly correlated with what you already own?
7. Do portfolios *have* to be negatively correlated or uncorrelated to offer the benefit of reduced risk through diversification?
8. A client of yours has a portfolio consisting of stocks that tend to move together. What would you recommend to her to help in reducing the risk of her portfolio?
9. Can you draw and label an investment opportunity set (IOS) of 2 risky securities?
10. How many different portfolios are there on the IOS that consists of 2 risky securities?
11. Where is the minimum variance portfolio on the IOS?
12. Is there a portfolio of "maximum variance" on the IOS?
13. Why shouldn't we always hold the minimum variance portfolio on the IOS if it has the lowest level of risk?
14. How does the IOS shape change as the correlation between assets increases and decreases?
15. Explain how the straight line IOS of perfectly correlated portfolios shows that overall portfolio risk increases when a riskier asset is added to the less risky asset.
16. What would be the shape of the IOS if two risky securities had a correlation coefficient of -0.9? What if they had a correlation of +0.9?
17. Do we expect that perfect positive and perfect negative correlations between different assets exist in practice?
18. A portfolio has an expected return of 11% and a standard deviation of 3%. Is this preferable to a portfolio that has an expected return of 10% and a standard deviation of 5%? What about relative to a portfolio that has an expected return of 11% and a standard deviation of 5%?
19. CHALLENGE Ten investors, each with identical levels of wealth, hold the *Vanguard S&P 500 Index Fund* and the *Vanguard Total Bond Market Index Fund*. They hold no other assets, not even cash or risk-free securities. Given these investors each hold the same two funds, is it possible for them to have different levels of risk? Why or why not?
20. CHALLENGE How would you define **efficient diversification**? What might be diversifying *inefficiently*?
21. CHALLENGE Inverse ETFs are funds that move in the opposite direction as a benchmark. For example, if the Vanguard S&P 500 ETF rises 1.8% in a day, the ProShares Short S&P 500 ETF will fall 1.8% that day. Why can't these seemingly *perfectly negatively* correlated funds be

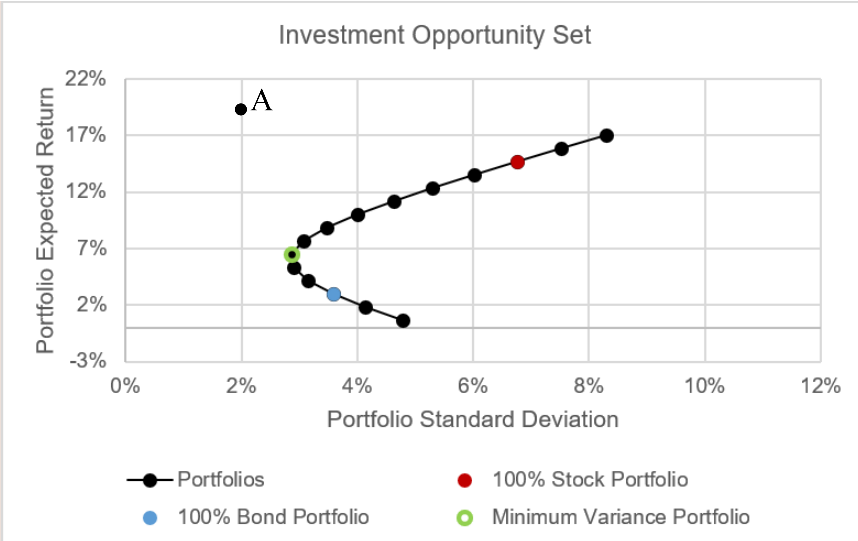
combined in such a way that their combination generates risk-free returns as shown in the top right diagram in Figure 2 of the lecture notes?

22. CHALLENGE Show how even a very high positive correlation between two assets offers a benefit when building a diversified portfolio. Hint: plug in $\rho = 0.99$ and $\rho = 1$ instead of 0.2 for the portfolio standard deviation problem in the lecture notes. Compare these results to a simple weighted average of the bond and stock funds' standard deviations of 8% and 19%.

ANALYTICAL QUESTIONS

Below are portfolios constructed of a stock fund and a bond fund. Use the output below to answer the questions that follow.

Portfolios			
w_S	w_B	$E(R)_P$	σ_P
120%	-20%	17.04%	8.3%
110%	-10%	15.87%	7.5%
100%	0%	14.70%	6.8%
90%	10%	13.52%	6.0%
80%	20%	12.35%	5.3%
70%	30%	11.18%	4.6%
60%	40%	10.01%	4.0%
50%	50%	8.83%	3.5%
40%	60%	7.66%	3.1%
30%	70%	6.49%	2.9%
20%	80%	5.32%	2.9%
10%	90%	4.14%	3.2%
0%	100%	2.97%	3.6%
-10%	110%	1.80%	4.1%
-20%	120%	0.63%	4.8%



1. What is the expected return and standard deviation of the stock fund?
2. What is the expected return and standard deviation of the bond fund?
3. If you wish to construct a portfolio of stocks and bonds that results in the lowest possible risk, how much of your money should you put in stocks and how much should you put in bonds?
4. What is the expected return and standard deviation of the portfolio with the lowest level of risk?
5. Explain if your answer to the question above is always true. What if the investor selects a different stock and bond fund?
6. Given the bond fund has a lower standard deviation than the stock fund, why won't holding 100% of your money in the bond portfolio minimize risk?
7. Explain why 17.04% is *not* the highest achievable return possible with the stock and bond fund.
8. The correlation between the stock and bond fund is -0.2156. If the correlation between the two were -0.99, how would the shape of the IOS change?
9. Explain why an investor *cannot* combine the stock and bond fund to achieve the point labeled "A" on the graph. Would an investor *want* the portfolio labeled "A" if it were possible to achieve?

Below is a table of correlations between common asset classes from 1972-2019 (Source: SBBI). Large (small) cap stocks have a large (small) market capitalization. Use the table to answer the questions that follow.

	Large-cap stocks	Small-cap stocks	Long-term corp. bonds	Long-term gov't bonds	Inter.-term gov't bonds
Small-cap stocks	0.72				
Long-term corp. bonds	0.28	0.13			
Long-term gov't bonds	0.04	-0.12	0.89		
Inter.-term gov't bonds	0.06	-0.04	0.83	0.87	
T-bills	0.04	0.05	0.05	0.10	0.43

1. Which two asset classes, based on correlation alone, would have offered the greatest diversification benefit over this period if the two were combined in a single portfolio?
2. Which two asset classes, based on correlation alone, would have offered the worst diversification benefit over this period if the two were combined in a single portfolio?
3. If you combined the two asset classes from the previous question into one portfolio, could the standard deviation of the resulting portfolio be lower than the standard deviation of either asset class individually?
4. Why do you suspect that government bonds and T-bills, despite both being issued by the federal government, have low correlations?
5. This data is from 1972-2019. Do we expect that correlations between assets will always remain the same? If correlations change, what does that mean for how we construct efficiently diversified portfolios?

CFA Questions

Answers are in the *Notes & References* section below.[3]

1. Two stocks have the same return and risk: 10% return with 20% standard deviation. You form a portfolio with 50% each in Stock 1 and Stock 2. What is the portfolio standard deviation if the correlation is 1.0?
    a. 20%
    b. 10%
    c. 5%
2. Two stocks have the same return and risk: 10% return with 20% standard deviation. You form a portfolio with 50% each in Stock 1 and Stock 2. What is the portfolio standard deviation if the correlation is 0.0?
    a. 26%
    b. 14%
    c. 15%
3. Two stocks have the same return and risk: 10% return with 20% standard deviation. You form a portfolio with 50% each in Stock 1 and Stock 2. What is the portfolio standard deviation if the correlation is -1.0?
    a. 0%
    b. 1%
    c. -1%

Notes & References

[1] We've used scenario analysis to arrive at estimates of variance and standard deviation. Alternatively, one might use historical data rather than using estimates of probabilities of various scenarios. That is, we assign equal probabilities to each outcome given each historically occurred. The standard deviation from scenario analysis:

$$\sigma = \sqrt{\sigma^2} = \sqrt{\sum_{s=1}^{S} p(s)[r(s) - E(r)]^2}$$

Becomes an *estimate* of the standard deviation $\boldsymbol{\hat{\sigma}}$ using historical data:

$$\hat{\sigma} = \sqrt{\widehat{\sigma^2}} = \sqrt{\frac{1}{S-1}\sum_{s=1}^{S}[\, r(s) - \bar{r}\,]^2}$$

Where S is, for example, the total number of periods and $\bar{r}$ is the average return of the asset over those periods. This is the `=STDEV.S()` formula for standard deviation in Excel. Covariance may also be estimated using historical data rather than scenario analysis. Rather than:

$$Cov(r_1, r_2) = \sum_{i=1}^{n} p(i)[r_1(i) - E(r_1)][r_2(i) - E(r_2)]$$

You can compute

$$\widehat{Cov(r_1, r_2)} = \frac{1}{n-1}\sum_{i=1}^{n}[r_1(i) - \bar{r_1}][r_2(i) - \bar{r_2}]$$

Where n is, for example, the total number of periods and $\bar{r}$ is the average return of that asset over those periods. This is the `=COVAR.S()` formula for standard deviation in Excel.

[2] Technically, one can solve for the exact weights of the two assets in the minimum variance portfolio. In the Excel file, I show the minimum variance portfolio of the *portfolio weightings displayed* and restrict the portfolios to 10 percentage-point weighting increments.

[3] CFA Question answers: 1)A, 2)B, 3)A