University of Richmond
Robins School of Business™

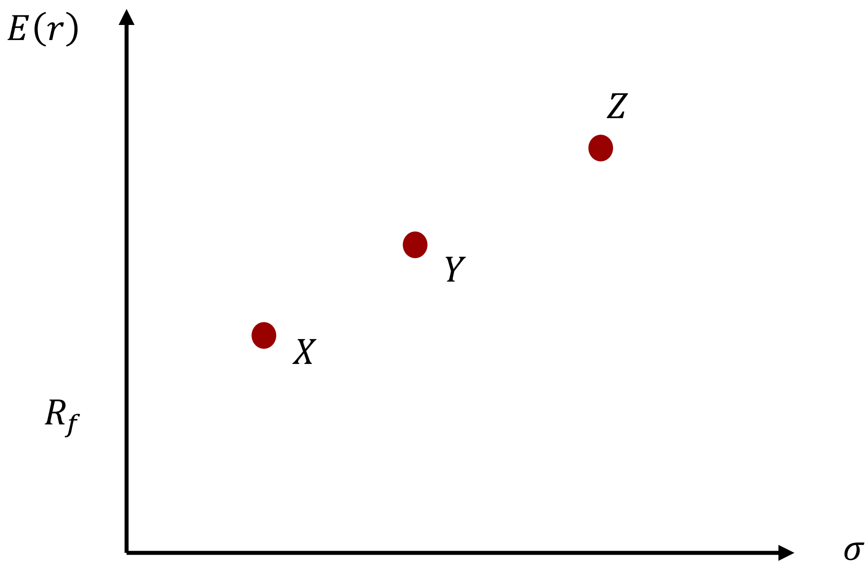
Many Risky Assets & the Index Model

Fin 366: Investments

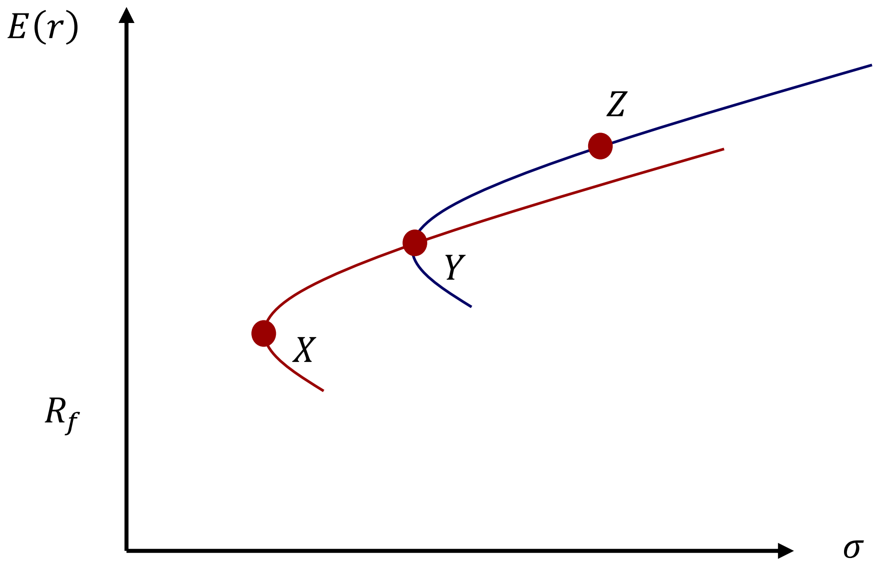
GENERALIZING THE TWO RISKY-ASSET APPROACH

We've seen how to construct optimal risky portfolios with two risky assets (for example, a stock fund and a bond fund). We then constructed a complete portfolio by supplementing the optimal risky portfolio with the risk-free asset.

Now, we generalize those results to *many* risky assets. Consider three risky assets X, Y, and Z that we graph on the return-risk axes.



We can draw two investment opportunity sets.



- The IOS connecting X and Y consists of different combinations of X and Y, obtained by varying the weights w_X and w_Y of each.

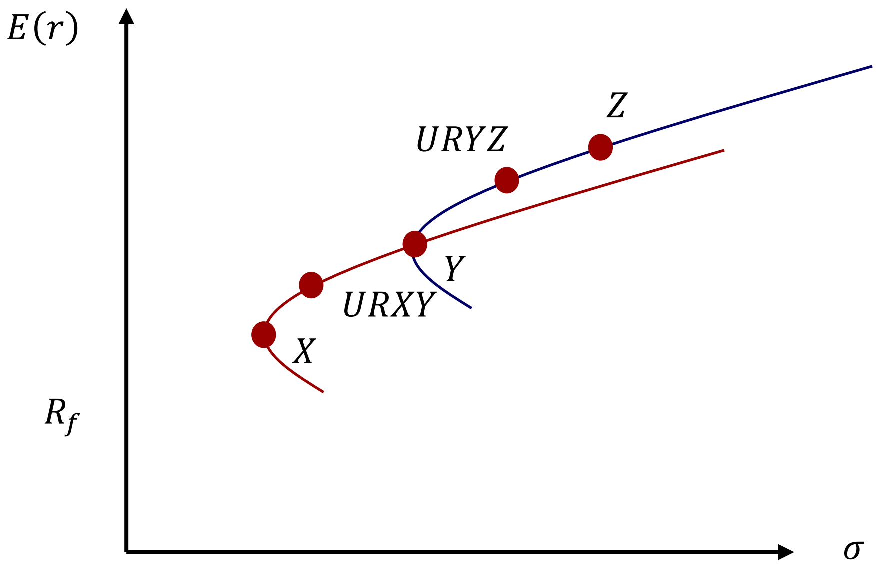
- The IOS connecting Y and Z consists of different combinations of Y and Z, obtained by varying the weights w_Y and w_Z of each.

Next, we construct portfolios of the assets XY and YZ. These "funds" will have arbitrary weights in the each of the assets. Let's call them URXY and URYZ, where URXY is a fund with some breakdown of X and Y while URYZ is a fund with some breakdown of Y and Z.

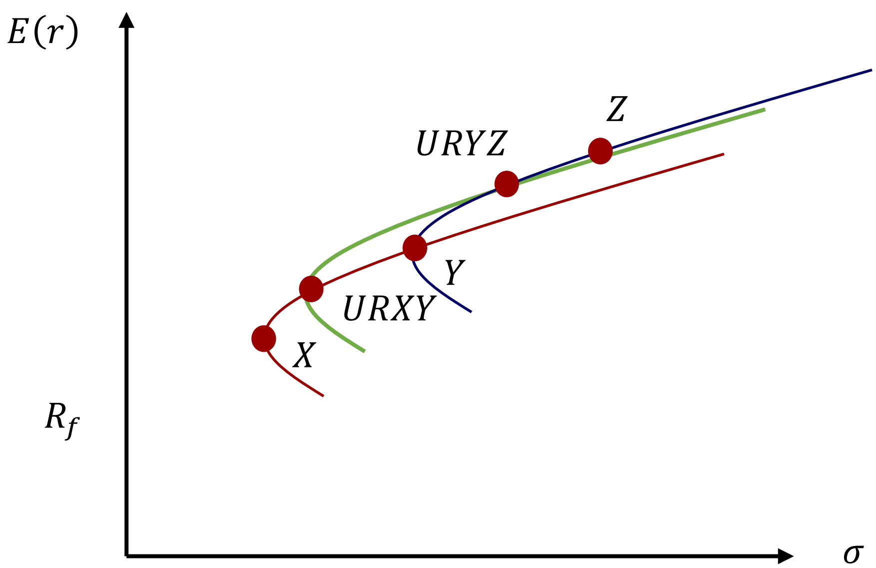
- Given URXY consists only of X and Y, it will lie on the IOS consisting of X and Y.

- Given URYZ consists of only Y and Z, it will lie on the IOS consisting of Y and Z.

Our figure becomes:

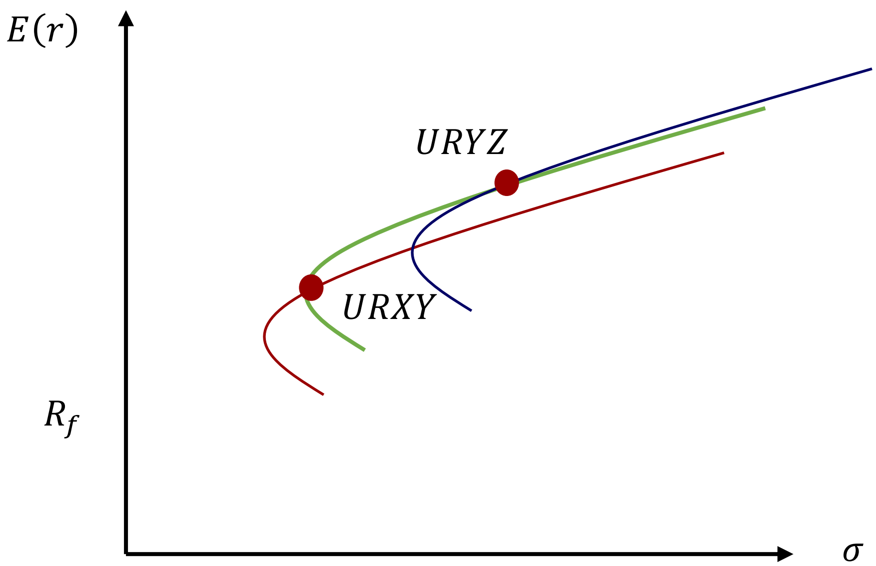


URXY and URYZ are risky assets, much like a bond fund or a stock fund in previous examples. We can similarly draw an IOS between them. It consists of all portfolios one can achieve by varying weights of investments in the funds URXY and URYZ.

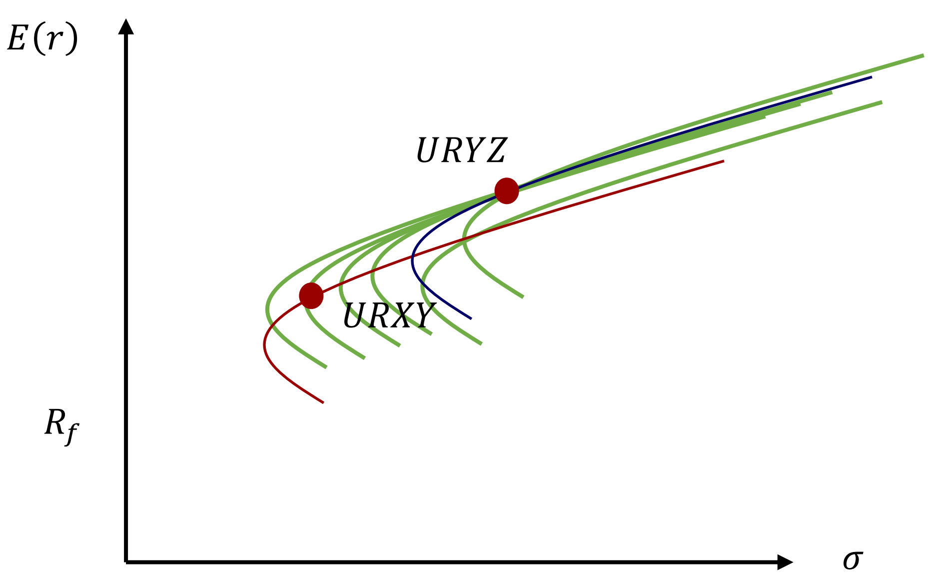


Given URXY consists of X and Y and URYZ consists of Y and Z*, the IOS connecting URXY and URYZ consists of all X, Y, and Z*.

Cleaning up the graph to consider only URXY and URYZ (which *includes* assets X, Y, and Z):

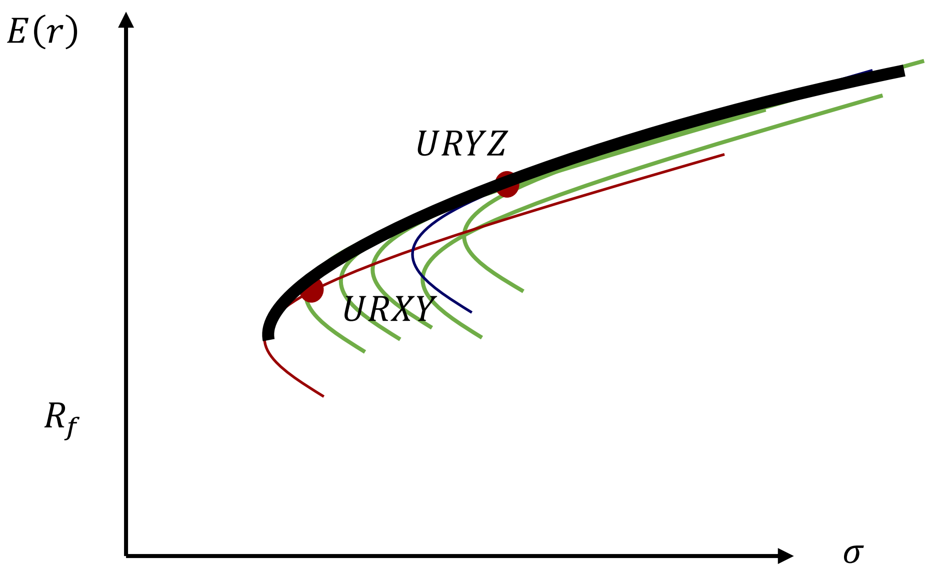


Multiple IOSs for URYZ and URXY are possible by varying the weights of X and Y in URXY and by varying the weights of Y and Z in URYZ.



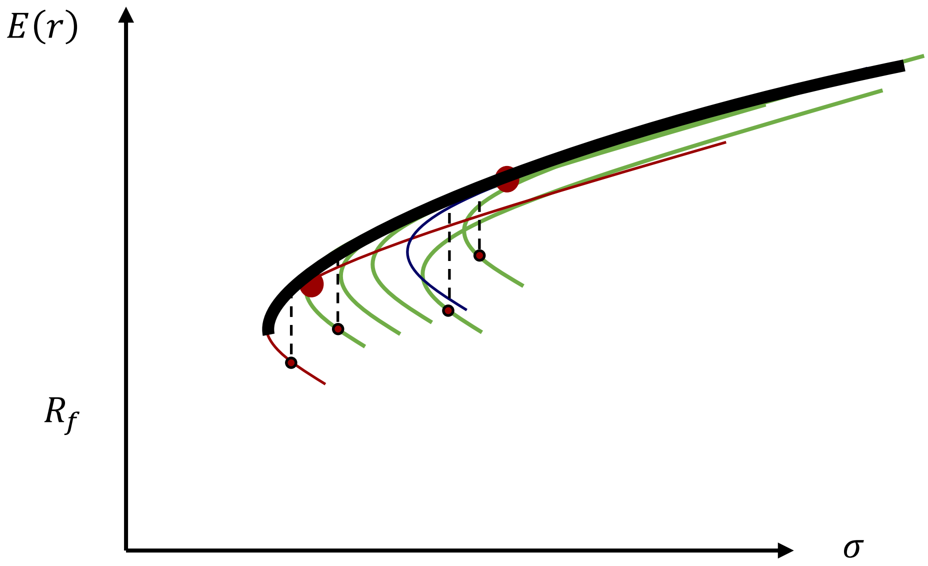
❓ Some of the IOSs drawn above don't appear to pass through the point for URXY or the point for URYZ. Why is this the case? Should this be possible?

There are infinitely many IOSs possible for URXY and URYZ, but not all are optimal. The **Efficient Frontier** consists of all those portfolios that maximize expected return at each level of risk. It connects all the "northwestern" most portfolios.



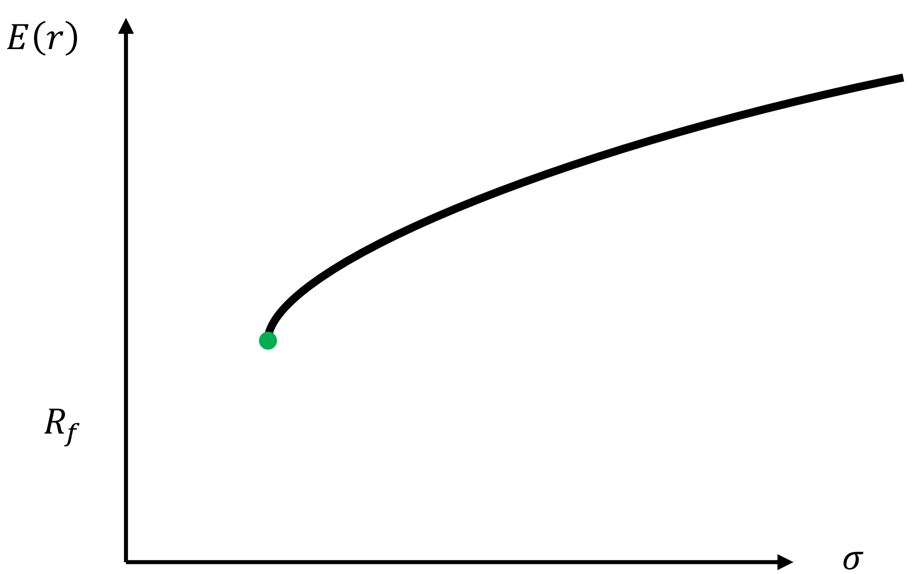
The **Efficient Frontier** is so called because all these portfolios that lie on the efficient frontier are "efficiently diversified". That is, they achieve maximum reward per unit of risk.

Portfolio combinations of URXY and URYZ *below* the efficient frontier achieve lower expected returns for that level of risk than they could achieve by simply adjusting the weights of URXY and URYZ.



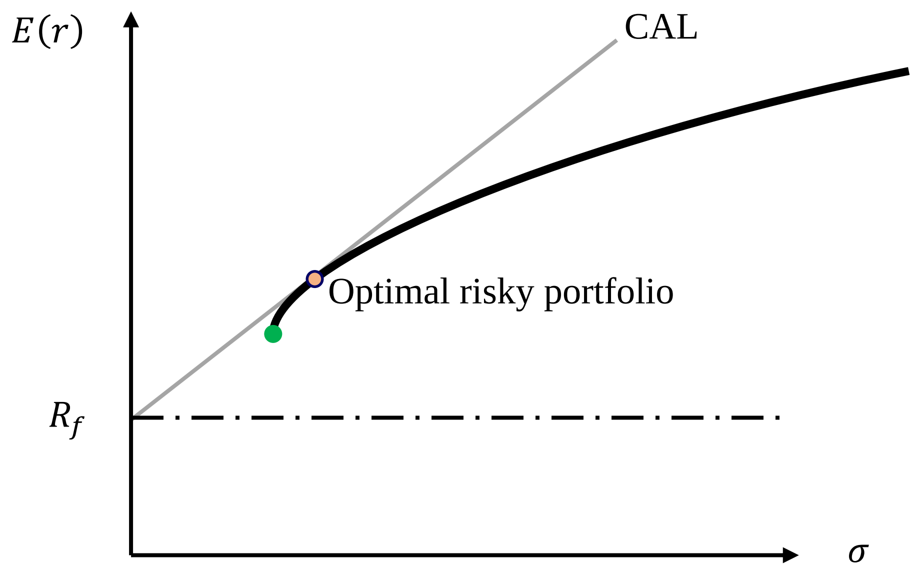
These portfolio combinations below the efficient frontier are *inefficient*. A portfolio with greater returns for that level of risk is available above it.

This efficient frontier begins at the **global minimum variance portfolio**, the portfolio with the lowest level of risk (and furthest to the left on the x-axis).



While the global minimum variance portfolio has the lowest risk, it is not necessarily the *optimal risky portfolio* that one should hold. As before, we determine the optimal risky portfolio at the point of tangency between the CAL drawn from the risk-free rate to the IOS.

In this case, the IOS is the efficient frontier. The CAL has the steepest slope (and therefore highest Sharpe ratio) possible given the feasible portfolio risk-return combinations.

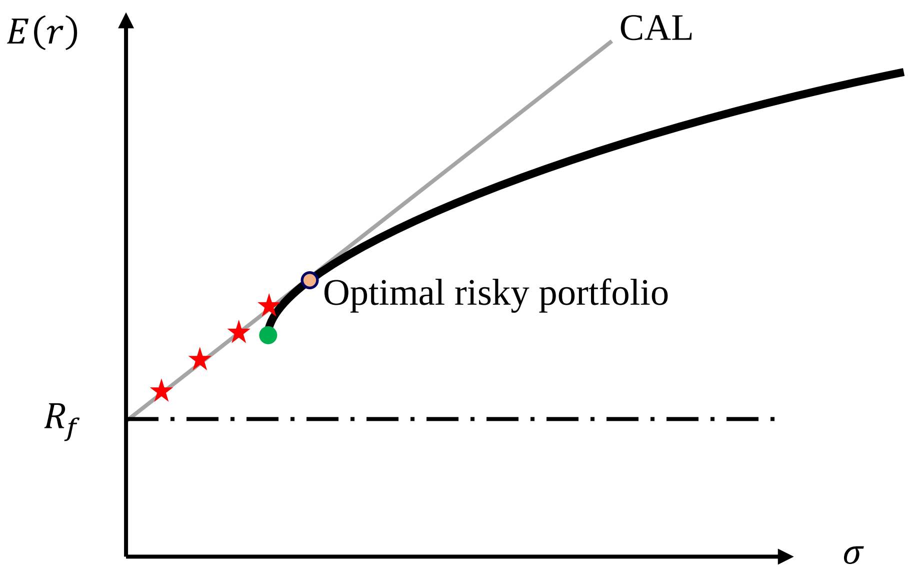


This optimal risky portfolio contains some weighting of X, Y, and Z through the URXY and URYZ combinations.

Can the global minimum variance portfolio ever be the optimal risky portfolio?

This principle expands beyond just three risky assets. We could repeat the illustrations for stocks W, X, Y, and Z, creating combos of URWX, URWY, URWZ, URXY, URXZ, and URYZ to develop an efficient frontier.

Lastly, one determines the **complete portfolio** on the CAL by choosing some weight to invest in the risk-free asset and some weight to invest in this optimal risky portfolio:



You can generate an efficient frontier for many risky asset classes on the website *Portfolio Visualizer*. Available at www.portfoliovisualizer.com/efficient-frontier.

OPTIMAL PORTFOLIOS IN PRACTICE

You ask 10 asset managers to compute the optimal risky portfolio for 2 stock funds and a bond fund that you specify. How many different optimal risky portfolio weights will you get? How many *should* you get?

We would expect *in theory* that asset managers would arrive at the same optimal portfolio for all their clients, and then make recommendations based on the weight they choose to invest in this risky portfolio vs. the risk-free asset.

In practice, different managers have different estimates of the inputs (expected returns, standard deviations, correlations, etc.). This results in deriving different efficient frontiers. Different "optimal risky" portfolios may be offered to different clients.

THE INDEX MODEL

COMPUTING OPTIMAL PORTFOLIOS IN PRACTICE

Recall the formula and inputs used to determine the optimal weights of assets in the two-risky asset scenario. To construct an efficient frontier from 50 securities, you'd have to make 1,325 calculations:

- 50 expected returns for the 50 securities
- 50 standard deviations for the 50 securities
- 1,225 covariances and correlations among the securities.

For 3,000 securities, you'd need more than 4.5 *million* estimates.[1] Obviously, this is a computationally intense problem handled by software. The Index Model we explore next offers useful simplifying estimations.

THE INDEX MODEL

Rather than produce the millions of computations required for expected returns, variances, and covariances to develop an optimal risky portfolio, we can recognize that positive covariates among security returns arise from common economic forces that affect the prospects of most firms.

The **Index Model** is a model relating stock returns to returns on both a broad market (systematic) and firm-specific (unsystematic) factors. We take the *overall market excess return* to be this index, and determine security "exposures" or "sensitivities" to this index.

We begin by graphically depicting the index model:

1. Plot an asset's excess returns (y-axis) relative to the market index's excess return (x-axis), where excess return is the return in excess of the risk-free rate

2. Determine the line of best fit (via regression) through the plot to describe the *typical* relationship between the return on the security and the return on the market

    - Slope: the security's *sensitivity* to the market return (**beta** or β)

    - y-intercept: the security's return when the market return is zero (**alpha** or α)

    - R^2: the portion of security's variation explained by the market

    - $(1-R^2)$: the portion of the security's risk that is firm specific

> ? How do these axes differ from what we've seen with the capital allocation line (CAL) and capital market line (CML)?

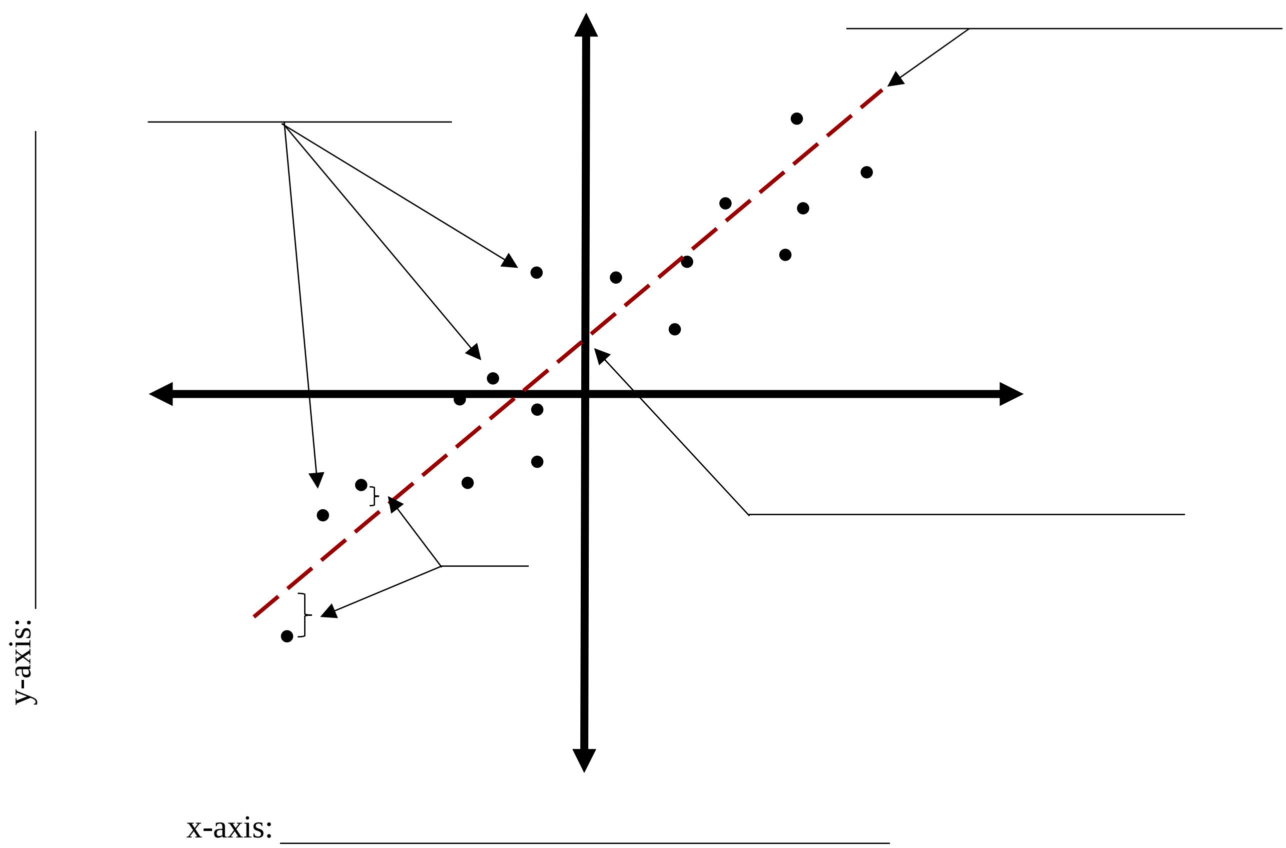
Given this relationship, the index model can be written as the following *regression equation*, following the standard $y = mx + b$ format. We regress the excess returns of the market on the excess returns of a security:

$$R_{i,t} = \alpha_i + \beta R_{M,t} + e_{i,t}$$

where the excess return *R* on security *i* in time *t* is linearly related to the excess return on the market *M* at time *t*. The error term *e* represents the firm-specific "surprise" in the security return in that time *t*.

The **Security Characteristic Line** is the line drawn through the scatterplot, with the above equation.

For further illustration, we will now use the Excel file *Index Model* available at josephfarizo.com/fin366.html. Below are axes for you to draw on and label.



Can a stock's beta be negative? Can alpha be negative? Can R-squared be negative?

END NOTES

[1] If $n = 3000$, then $(n^2 - n)/2 = 4{,}498{,}500$ covariances, 3000 expected returns, and 3000 standard deviations.