§10. Optimal Risky and the Complete Portfolio

FIN 366: Investments

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The Two Risky Asset Complete Portfolio

Recall the **complete portfolio** consisting of a risky portion and a risk-free portion. Now, we consider two risky assets *within* the risky portion, a stock fund *S* and a bond fund *B*. These will be held along with the risk-free portion, T-bills.

Figure 1: The Complete Portfolio

Complete Portfolio = $100		
Risky Portfolio (30%)		Risk-free Portfolio (70%)
Mutual Funds ($30)		*T-Bills ($70)*
S Fund ($10)	*B Fund ($20)*	*T-Bills ($70)*

Complete Portfolio = $100		
Risky Portfolio (60%)		Risk-free Portfolio (40%)
Mutual Funds ($60)		*T-Bills ($40)*
S Fund ($20)	*B Fund ($40)*	*T-Bills ($40)*

Optimal Risky Portfolio Weights

We can, by *computation*, arrive at the **optimal risky portfolio**, or the appropriate weights of the stock fund and the bond fund in the risky portfolio that results in the *best achievable return-risk* combination given their characteristics (return, risk, correlation).

We then, by *choice*, arrive at the **complete portfolio** by varying how much we invest in the (1) optimal risky portfolio and (2) the risk-free portfolio.

The Separation Property

The **separation property** defines this two-step process of portfolio construction.

Portfolio construction can be separated into 2 independent tasks. The **complete portfolio** is found by:

(1) *Calculation* of the optimal risky portfolio, and
(2) *Choice*, based on the risk tolerance of the investor, of the mix of the optimal risky portfolio and the risk-free asset.

Computing Risky Asset Weights

To calculate the appropriate weights of the stock (1) and bond (2) funds in the risky portfolio, we compute:

$$w_1 = \frac{[E(r_1) - r_f]\sigma_2^2 - [E(r_2) - r_f]\sigma_1\sigma_2\rho_{1,2}}{[E(r_1) - r_f]\sigma_2^2 + [E(r_2) - r_f]\sigma_1^2 - [E(r_1) - r_f + E(r_2) - r_f]\sigma_1\sigma_2\rho_{1,2}}$$

and

$$w_2 = 1 - w_1$$

These formulas are determined by an optimization problem using standard tools of calculus.[1] This calculation will result in the portfolio with the *highest reward per unit of risk*¸ or the best achievable Sharpe ratio. This process of maximizing returns per unit of risk is known as **mean-variance optimization**.

PRACTICE: You'd like to construct a complete portfolio consisting of the Fidelity US Bond Index Fund (FXNAX), the Fidelity S&P 500 Index Fund (FXAIX), and T-Bills. Determine the (1) weights of the optimal risky portfolio consisting of the two risky funds, (2) the expected return of the optimal risky portfolio, and (3) the standard deviation of the optimal risky portfolio given the following inputs:

- The yield on a risk-free money market fund investing in T-Bills is currently 1.0%
- The expected return of the FXNAX is 6.6%, with a standard deviation of 30%
- The expected return of the FXAIX is 19.1%, with a standard deviation of 43%
- The correlation coefficient between these funds is -0.15

Then, assuming you choose to invest 57% in the risky portfolio given your personal risk preferences, (4) determine the expected return, risk, and Sharpe ratio of the *complete* portfolio.

SOLUTION: Let $w_1 = w_B$ be the weight of the bond fund FXNAX in the optimal risky portfolio and let $w_2 = w_S$ be the weight of the stock fund FXAIX in the optimal risky portfolio. Plugging the inputs into the formula for the optimal weights above yields:

$$w_1 = \frac{[E(r_1) - r_f]\sigma_2^2 - [E(r_2) - r_f]\sigma_1\sigma_2\rho_{1,2}}{[E(r_1) - r_f]\sigma_2^2 + [E(r_2) - r_f]\sigma_1^2 - [E(r_1) - r_f + E(r_2) - r_f]\sigma_1\sigma_2\rho_{1,2}}$$

$$w_B = \frac{[E(r_B) - r_f]\sigma_S^2 - [E(r_S) - r_f]\sigma_B\sigma_S\rho_{BS}}{[E(r_B) - r_f]\sigma_S^2 + [E(r_S) - r_f]\sigma_B^2 - [E(r_B) - r_f + E(r_S) - r_f]\sigma_B\sigma_S\rho_{BS}}$$

$$= \frac{[0.066 - 0.01]0.43^2 - [0.191 - 0.01](0.30)(0.43)(-0.15)}{[0.066 - 0.01]0.43^2 + [0.191 - 0.01]0.30^2 - [0.066 - 0.01 + 0.191 - 0.01](0.30)(0.43)(-0.15)}$$

$$w_B = 0.444$$

Therefore:

$$w_S = 1 - 0.444 = 0.556$$

INTERPRETATION: We now have our weights in the optimal risky portfolio: 44.44% in the bond fund FXNAX and 55.6% in the stock fund FXAIX. Given these weights, we can determine the expected return, risk, and Sharpe ratio of the optimal risky portfolio next using the formulas we already know. Let's denote the risky portfolio as "*p*" in the subscripts.

The expected return is:

$$E(r_p) = w_1E(r_1) + w_2E(r_2) = 0.444(0.066) + 0.556(0.191) = 13.6\%$$

The standard deviation is:

$$\sigma_p = \sqrt{(w_1\sigma_1)^2 + (w_2\sigma_2)^2 + 2(w_1\sigma_1)(w_2\sigma_2)\rho_{1,2}}$$

$$= \sqrt{(0.44 \times 0.30)^2 + (0.556 \times 0.43)^2 + 2(0.44 \times 0.30)(0.556 \times 0.43)(-0.15)}$$

$$= 0.256 = 25.6\%$$

The Sharpe ratio is:

$$S_P = \frac{E(r_P) - r_f}{\sigma_P} = \frac{0.136 - 0.01}{0.256} = 0.492$$

INTERPRETATION: Summarizing what we have so far:

- In the optimal risky portfolio, 44.4% should be in the bond fund FXNAX
- In the optimal risky portfolio, 55.6% should be in the stock fund FXAIX
- Given those weights, the expected return of the **risky portfolio** is 13.6%
- Given those weights, the standard deviation of the **risky portfolio** is 25.6%
- Given those weights, the Sharpe ratio of the **risky portfolio** is 0.492
- From the problem, we will invest 57% of our money in this risky portfolio
- From the problem, the risk-free asset yields 1%

Now, we determine the expected return, standard deviation, and Sharpe ratio of the **complete portfolio**. We know from the problem that 57% of our money will be in the optimal risky portfolio and $1 - 0.57 = 43\%$ in the risk-free asset. Now, for the complete portfolio:

$$E(r_C) = yE(r_P) + (1 - y)r_f = 0.57(0.136) + (1 - 0.57)(0.01) = 0.0818$$

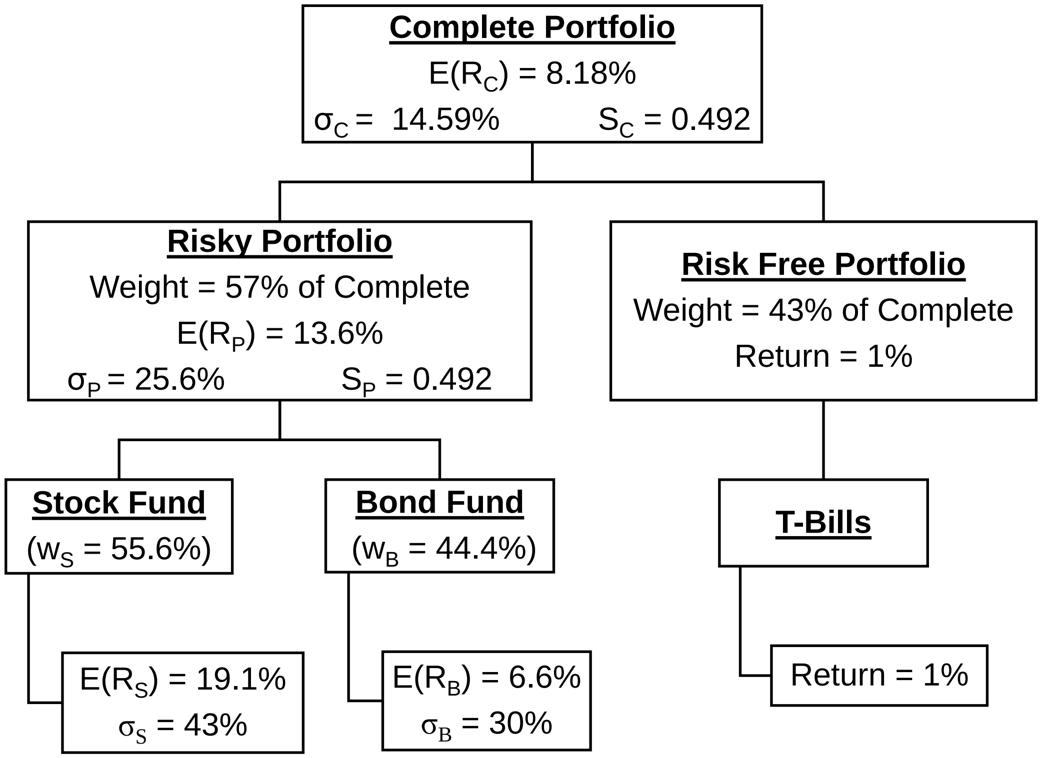
$$= 8.18\%$$

$$\sigma_C = y\sigma_{OptimalRisky} = 0.57 \times 0.256 = 0.1459 = 14.59\%$$

$$S_C = \frac{E(r_C) - r_f}{\sigma_C} = \frac{0.0818 - 0.01}{0.1459} = 0.492$$

The Sharpe ratio for the complete portfolio is the same as the Sharpe ratio for the optimal risky portfolio. Changing the weight *y* in the risky portfolio changes the risk and return, but it does not change the Sharpe ratio.

INTERPRETATION: We can finish this problem by summarizing all that we've found:



And finally, given the weights of the stock and bond fund in the risky portfolio, we can find the weights of each asset in the complete portfolio:

Complete Portfolio Weights:

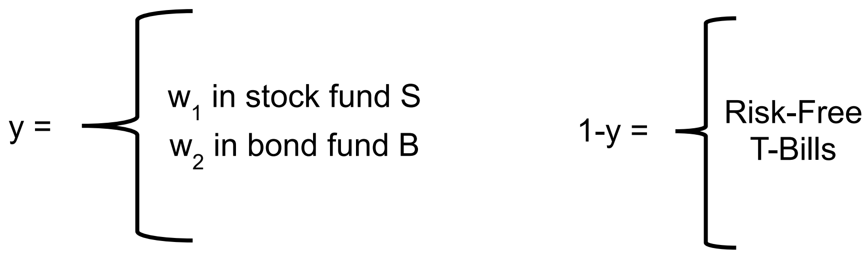
- Hold $0.556 \times 0.57 = 31.69\%$ in the Stock Fund (FXAIX)
- Hold $0.444 \times 0.57 = 25.31\%$ in the Bond Fund (FXNAX)
- Hold 43% in the risk-free asset

And we confirm that $31.69\% + 25.31\% + 43\% = 100\%$ of the complete portfolio.

THE CAL AND THE COMPLETE PORTFOLIO

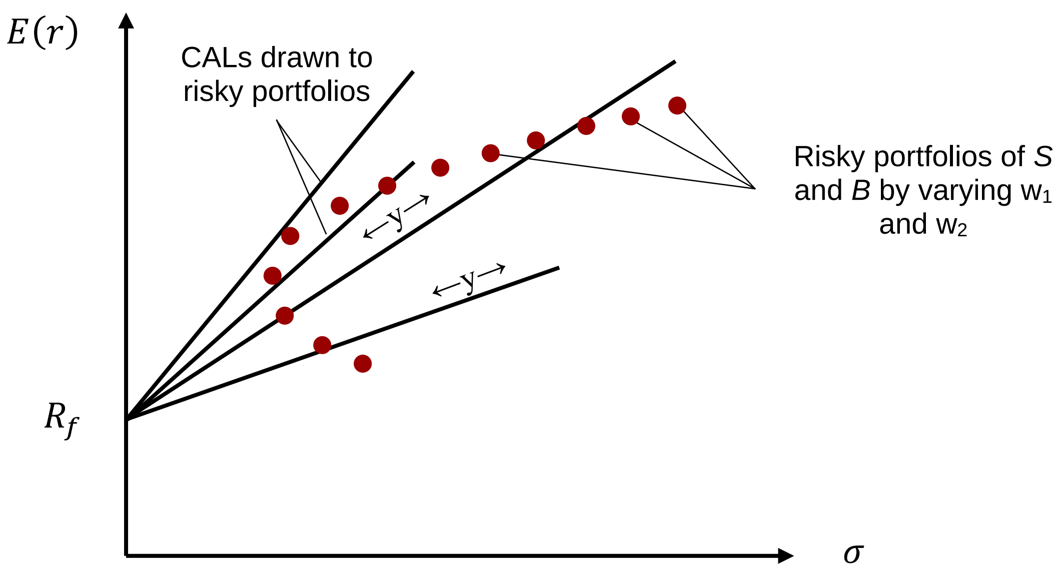
We can additionally use the **Capital Allocation Line** (**CAL**) to identify the optimal risky portfolio on an investment opportunity set. Recall the CAL is drawn from the risk-free rate of return to a risky portfolio. y is the weight in the risky asset and $1-y$ is the weight in the risk-free asset. Previously, we took the "risky asset" to be one asset. Now, we decompose this risky portfolio into the stock fund S and bond fund B components:

Figure 2: Capital Allocation and the Complete Portfolio



By the separation property, *y* and *(1-y)* are chosen. w_1 and w_2 are calculated.

Figure 3: The CAL and IOS of 2 Risky Assets



Given a risk-free rate (the yield on 90-day T-bills) we can overlay multiple CALs and the IOS of two risky assets because the CAL may be drawn from the risk-free rate to *any* risky portfolio on the return-risk axes.

From our previous discussion regarding the CAL, we know the *optimal portfolio maximizes return per unit of risk or has the greatest Sharpe ratio*. Therefore, we find the optimal risky portfolio (the weights in *S* and *B*) by identifying the CAL that:

(1) Is *tangent* to the Investment Opportunity Set

(2) With the steepest slope (highest Sharpe)

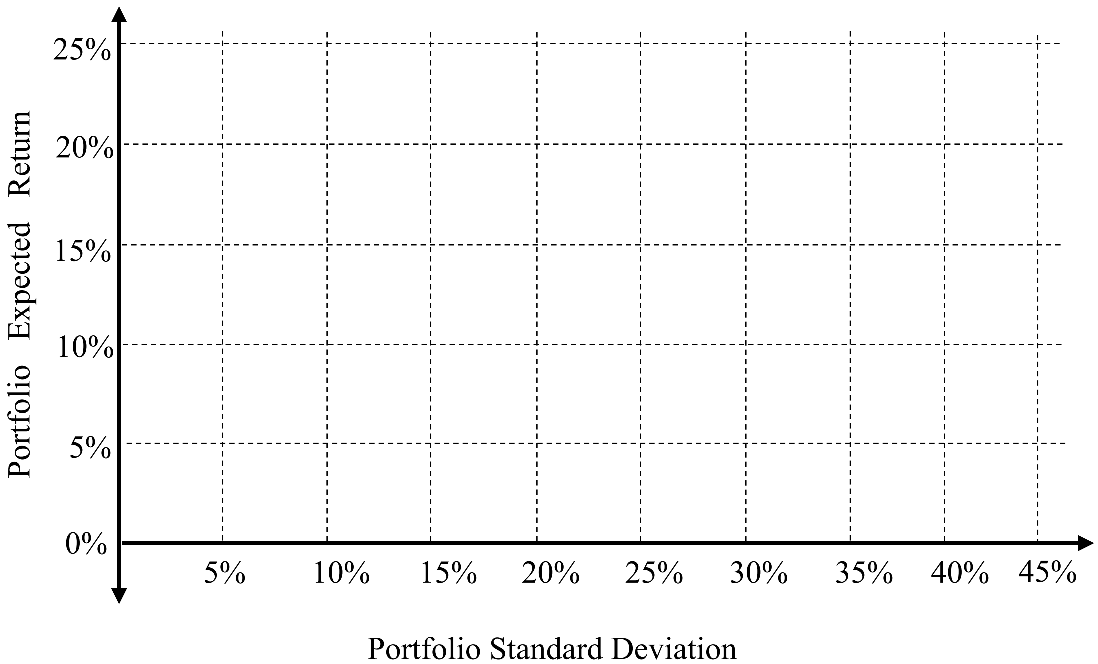
Which of the CALs in the figure above identifies the optimal risky portfolio?

View the Excel file *Optimal Risky Portfolio* at josephfarizo.com/fin366.html for additional illustrations and examples of different IOSs, optimal risky portfolios, and CALs.

PRACTICE: Using the same *S* and *B* fund in the previous example, draw the CAL that identifies the optimal risky portfolio. Show where on this CAL the *complete* portfolio lies.

w_S	w_B	$E(R_P)$	σ_P
100%	0%	19.1%	43.0%
80%	20%	16.6%	34.0%
60%	40%	14.1%	26.8%
40%	60%	11.6%	23.0%
20%	80%	9.1%	24.2%
0%	100%	6.6%	30.0%

SOLUTION: On the return-risk axes, and recalling $E(R_P)$ = 13.6%, σ_P = 25.6%, w_S = 55.6%, w_B = 44.4%, $E(R_C)$ = 8.18%, σ_C = 14.59%, and S_C = 0.492 while the risk-free rate is 1%. We choose 57% in the risky portfolio and 43% in the risk-free.



View the Excel file *Complete Portfolio* at josephfarizo.com/fin366.html for additional practice problems and a step-by-step guide to calculating optimal risky portfolios, complete portfolios, and drawing the IOS with the appropriate CAL.

This Excel file will walk you through a complete example encompassing this entire section. All questions are randomized, so you will never work the same problem twice.

Generalizing the Two Risky Asset Portfolio

We've seen how to construct optimal risky portfolios with two risky assets (for example, a stock fund and a bond fund). We then constructed a complete portfolio by supplementing the optimal risky portfolio with the risk-free asset.

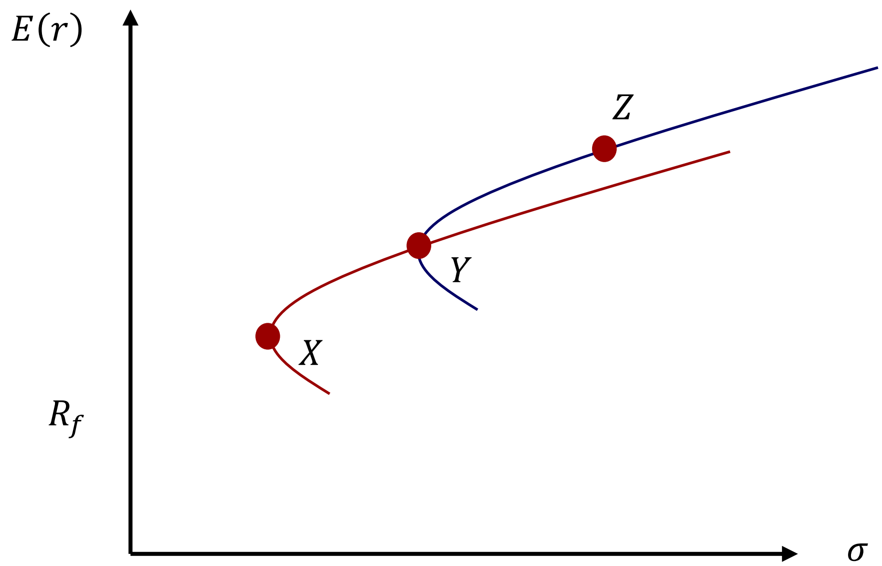
Three Risky Assets

Now, we generalize those results to *many* risky assets. Consider three risky assets X, Y, and Z that we graph on the return-risk axes. We can draw two IOSs:

(1) The IOS connecting X and Y consists of different combinations of X and Y, obtained by varying the weights w_X and w_Y of each.

(2) The IOS connecting Y and Z consists of different combinations of Y and Z, obtained by varying the weights w_Y and w_Z of each.

Figure 4: Three Risky Assets



CREATING MULTI-ASSET PORTFOLIOS

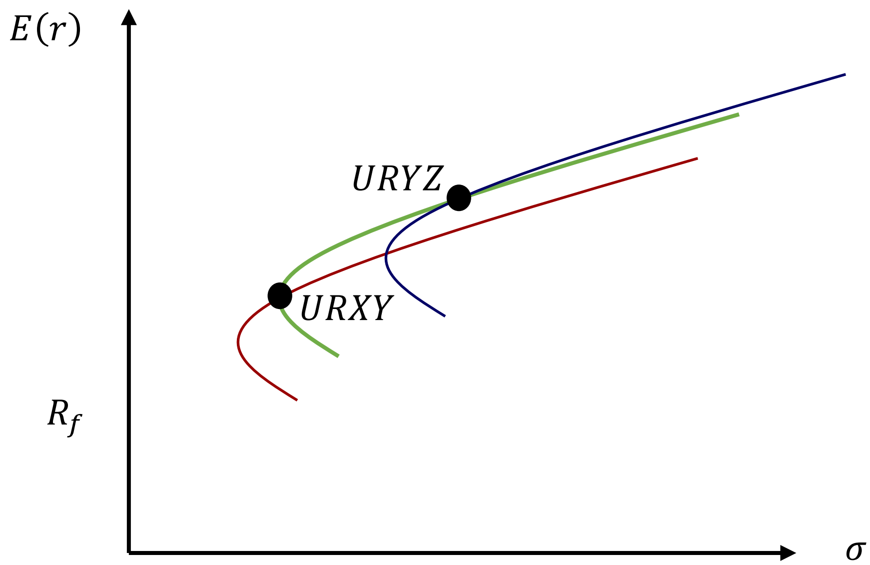
Next, we construct portfolios of the assets XY and YZ. These “funds” will have arbitrary weights in each of the assets. Let’s call them URXY and URYZ, where URXY is a fund with some weightings of X and Y while URYZ is a fund with some weightings of Y and Z.

- Given URXY consists only of X and Y, it will lie on the IOS consisting of X and Y.

- Given URYZ consists of only Y and Z, it will lie on the IOS consisting of Y and Z.

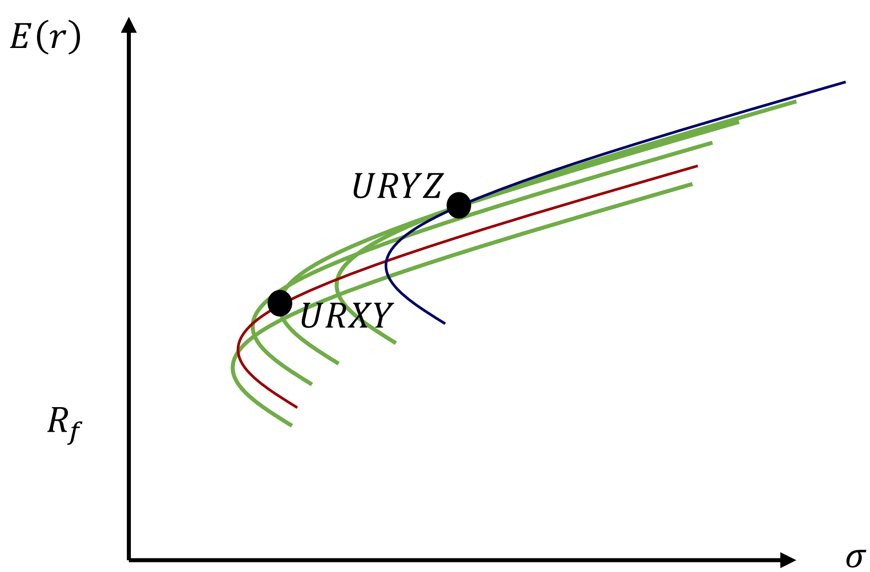
URXY and URYZ are risky assets, much like a bond or stock fund in previous examples. We can similarly draw an IOS between them. It consists of all portfolios one can achieve by varying weights of investments in the funds URXY and URYZ. This IOS contains assets X, Y, *and* Z.

Figure 5: IOS of 3 Risky Assets



Multiple IOSs for URYZ and URXY are possible by varying the weights of X and Y in URXY and by varying the weights of Y and Z in URYZ.

Figure 6: Multiple IOSs by Varying Weights X, Y, and Z

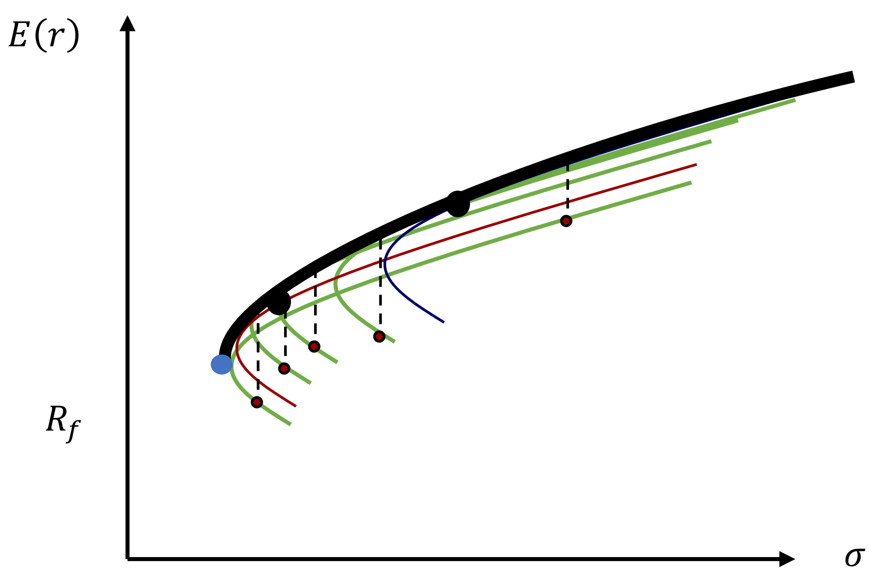


THE EFFICIENT FRONTIER

There are infinitely many IOSs possible for URXY and URYZ, but not all are optimal. The **Efficient Frontier** consists of all those portfolios that maximize expected return at each level of risk. It connects all the "northwestern" most portfolios. The **Efficient Frontier** is so called because all portfolios that lie on the efficient frontier are "efficiently diversified". That is, they achieve maximum reward per unit of risk for their risk level.

Portfolio combinations of URXY and URYZ *below* the efficient frontier achieve lower expected returns for that level of risk than they could achieve by simply adjusting the weights of URXY and URYZ. Points above the efficient frontier are *infeasible* and can't be obtained by adjusting weights.

Figure 7: The Efficient Frontier



The portfolio combinations below the efficient frontier are *inefficient*. A portfolio with greater returns for that level of risk is available above it simply by adjusting weights.

This efficient frontier begins at the **global minimum variance portfolio**, the portfolio with the lowest level of risk (and furthest to the left on the x-axis). While the global minimum variance portfolio has the lowest risk, it is not necessarily the *optimal risky portfolio* that one should hold.

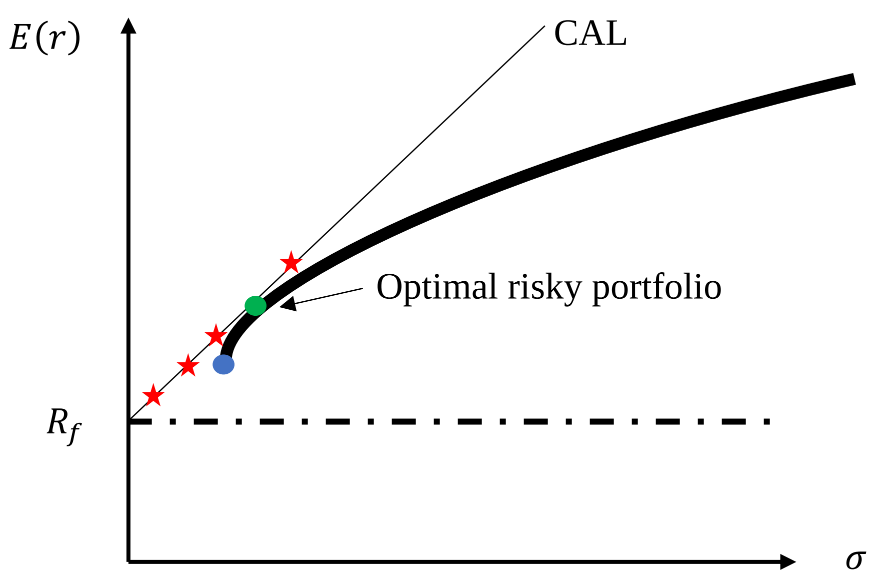
As before, we determine the optimal risky portfolio at the point of tangency between the CAL drawn from the risk-free rate to the IOS. In this case, the IOS is the efficient frontier. The CAL has the steepest slope (and therefore highest Sharpe ratio) possible given the feasible portfolio risk-return combinations.

This optimal risky portfolio contains some weighting of X, Y, and Z through the URXY and URYZ combinations. This principle expands beyond just three risky assets. We could repeat the illustrations for stocks W, X, Y, and Z, creating combos of URWX, URWY, URWZ, URXY, URXZ, and URYZ to develop an efficient frontier.

The Complete Portfolio

Lastly, one determines their **complete portfolio** ★ on the CAL by choosing some weight to invest in the risk-free asset and some weight to invest in this optimal risky portfolio. All complete portfolios have the same Sharpe, given they lie on the same CAL:

Figure 8: The Complete Portfolio



You can generate an efficient frontier for many risky asset classes on the website *Portfolio Visualizer*. Available at www.portfoliovisualizer.com/efficient-frontier.

Optimal Risky Portfolios in Practice

We would expect *in theory* that asset managers would arrive at the same optimal risky portfolio for all their clients, and then make recommendations based on the weight they choose to invest in this risky portfolio versus the risk-free asset.

In practice, different managers have different estimates of the inputs (expected returns, standard deviations, correlations, etc.). This results in their deriving different efficient frontiers. Different optimal risky portfolios may therefore be offered to different clients.

CRITICAL THINKING QUESTIONS

1. Describe the two steps of portfolio construction defined by the separation property.
2. Why in theory is the optimal risky portfolio of two assets the same for everyone who holds those two assets, regardless of how risk averse they are?
3. Assume you calculate the optimal weights between a risky stock fund FCNTX and a risky bond fund FCBFX to be 80%/20%. How do you explain this breakdown to someone who prefers a more conservative portfolio?
4. You calculate that the optimal weights for the ABC stock fund and XYZ bond funds are 82% ABC and 18% XYZ. Does that mean the optimal weights for the ABC stock fund and the UVW bond fund are also 82% and 18%?
5. How are the optimal risky asset weights determined? (Hint: look to the end notes of this section. Answer in *very broad* terms, no need to be overly technical).
6. Why would choosing to hold risky assets with weights different than what we compute be suboptimal?
7. Why is choosing *any* combination of the optimal risky portfolio and the risk-free asset as good as any other combination of the optimal risky portfolio and the risk-free asset?
8. Why are the correlations between two risky assets an important input in determining the optimal weights in the risky portfolio?
9. How can you determine graphically where the optimal risky portfolio is on an investment opportunity set of two risky assets?
10. How is it possible for the complete portfolio to be off the investment opportunity set of two risky assets? Will it always be off this investment opportunity set?
11. What will happen to the weights we hold in two risky assets as the expected returns, standard deviations, and correlations change through time?
12. Are all IOSs an efficient frontier? Are all efficient frontiers an IOS?
13. Why should an investor avoid holding a portfolio below the efficient frontier?
14. How is it possible in the first place to hold a security below the efficient frontier? Don't portfolios have to fall on an IOS for them to be feasible?
15. Why is a portfolio to the northwest of the efficient frontier not feasible?
16. Are all portfolios on the efficient frontier *optimal*? (Hint: consider drawing a CAL from the risk-free rate to the efficient frontier.)
17. Must the complete portfolio that consists of the optimal risky portfolio and the risk-free asset always fall on the efficient frontier?
18. Must the complete portfolio that consists of the optimal risky portfolio and the risk-free asset always fall on the tangent CAL?
19. Why in practice can different portfolio managers and investors develop different optimal risky portfolios?
20. Why do we need to compute a high number of covariances (and therefore correlations) when we increase our risky portfolio to include many risky assets?
21. You have a net worth of $100,000. You hold three mutual funds with tickers VANF, FIDL, and SSTR, with $30,000 in each. The rest of your wealth you hold in cash. A financial advisor tells you "I have computed the optimal weights to hold in your 3 funds. Rather than hold 33.33%

of your risky portfolio in each, you should hold 50% of your ‘risky wealth’ in VANF, 25% in FIDL, and 25% in SSTR. Additionally, to optimize your complete portfolio, you should hold less cash and invest more in these three funds.” Critique this advisor’s advice.

22. Assume instead the advisor in the previous question says: “I have computed the optimal weights to hold in your 3 funds. Rather than hold 33.33% of your risky portfolio in each, you should hold 50% of your ‘risky wealth’ in VANF, 25% in FIDL, and 25% in SSTR. Additionally, you can achieve greater expected returns if you hold less cash and invest more in these three funds.” Is this better advice relative to the previous question?
23. CHALLENGE Describe the difference between the (1) investment opportunity set created between the risk-free asset and a risky portfolio and the (2) investment opportunity set created between two risky assets. What do the shapes of these different “IOSs” generally look like? How does an investor move along these different IOSs?
24. CHALLENGE In our examples, we’ve considered a stock fund and a bond fund as our two risky assets. Assume that the stock and bond funds are both actively managed. In broad terms, describe what happens to our illustrations of the IOS of risky portfolios and the capital allocation line as the fund managers change the composition of their funds.
25. CHALLENGE Suppose you have 5 funds that you’d like to hold in optimal weights such that your Sharpe ratio is maximized. Is the optimal portfolio *guaranteed* to hold all 5 funds to some degree, or might some of the 5 funds be omitted from the optimal risky portfolio? Consider using the online Portfolio Visualizer tool from the lecture notes and your own choice of 5 risky assets to answer this question.

ANALYTICAL QUESTIONS

1. Below is an output from portfoliovisualizer.com that presents the mean-variance optimized allocations for a portfolio of (1) *US Stocks*, (2) *Corporate bonds*, and (3) the *Global ex-US Stock Market*. The *Global ex-US Stock Market* portfolio consists of stock in non-US firms. The data for these three **asset classes** is from 2003 to 2024. Use the output to answer the questions that follow.

Efficient Frontier Assets

#	Asset	Expected Return	Standard Deviation	Sharpe Ratio	Min. Weight	Max. Weight
1	US Stock Market	11.50%	15.23%	0.662	0.00%	100.00%
2	Global ex-US Stock Market	8.35%	17.21%	0.403	0.00%	100.00%
3	Corporate Bonds	4.44%	8.08%	0.373	0.00%	100.00%

Results based on historical returns. Expected return is the annualized monthly arithmetic mean return. Ex-ante Sharpe Ratio calculated using 3-month treasury bill returns as the risk-free rate.

Asset Correlations

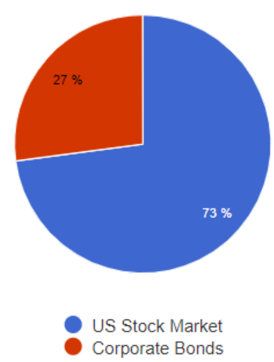
Asset	US Stock Market	Global ex-US Stock Market	Corporate Bonds
US Stock Market	1.00	0.88	0.41
Global ex-US Stock Market	0.88	1.00	0.47
Corporate Bonds	0.41	0.47	1.00

Based on monthly returns from Jan 2003 to Feb 2024

Max Sharpe Ratio Portfolio

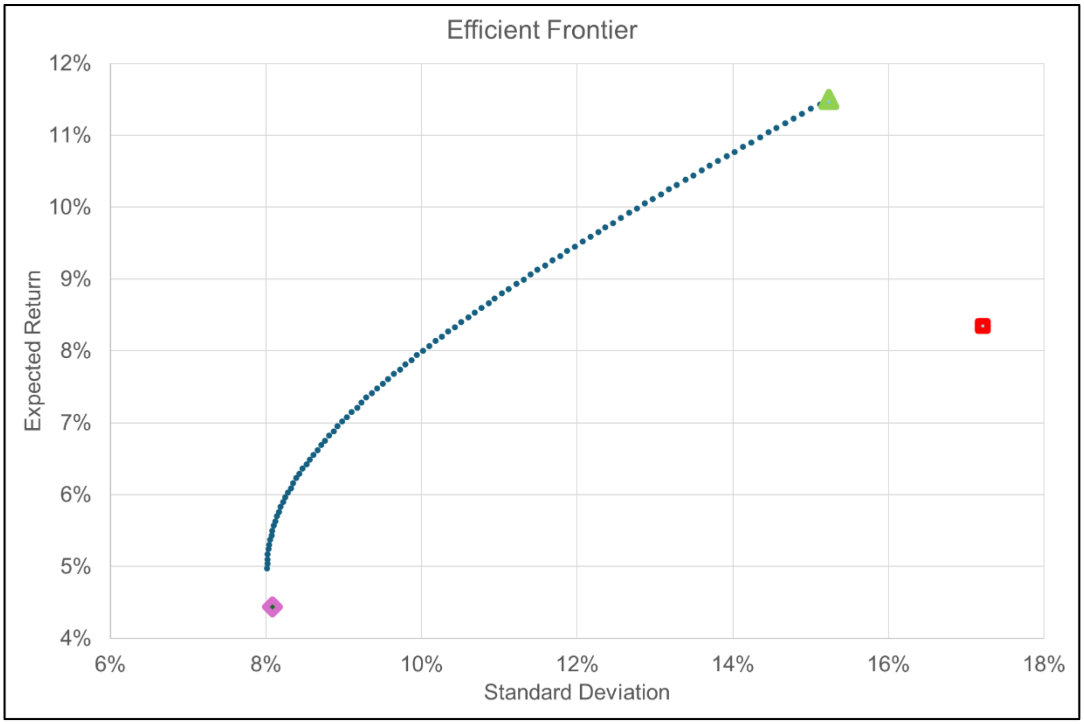
Asset Class	Allocation
US Stock Market	72.92%
Corporate Bonds	27.08%

Save portfolio »



Performance Summary

Portfolio	Expected Return	Standard Deviation	Sharpe Ratio
Max Sharpe Ratio Portfolio	9.59%	12.17%	0.67



a. In the figure, label (1) the *US Stocks* portfolio, the (2) *Corporate bonds* portfolio, and (3) the *Global ex-US Stock Market* portfolio.
b. In the figure, plot and label the optimal risky portfolio. How much of each asset class is represented in the optimal risky portfolio?
c. What is the approximate risk and return of the minimum variance portfolio? Where is it on the graph? Is this the "best" risky portfolio an investor should hold?
d. Looking to the correlations and Sharpe ratios of the asset classes, why do you suspect that the *Global ex-US stock Market* portfolio is not a part of the optimal risky portfolio?
e. If an investor chooses to hold the optimal risky portfolio, how would you recommend they allocate their capital between the optimal risky portfolio and the risk-free asset?
f. Why are risk-return combinations of points above and to the left of the efficient frontier not achievable, but the risk-return combinations of points below and to the right of the efficient frontier are achievable?
g. What could cause the weightings in the optimal risky portfolio to change?

2. Below is the image from the cover of Janus Henderson's "Trends and Opportunities" report. What does this image show? Explain as if you are describing the image to someone who is not trained in modern portfolio theory.

CFA QUESTIONS

Answers are in the *Notes & References* section below.[2]

1. Portfolios are *most likely* to provide:
    a. Risk reduction
    b. Risk elimination
    c. Downside protection
2. The minimum variance portfolio constructed of a set of risky assets
    a. Has the highest achievable Sharpe ratio
    b. Must consist of some weighting of all risky assets
    c. Has the lowest achievable standard deviation when combining the risky assets
3. *Inefficient* portfolios
    a. Lie to the northwest of the efficient frontier
    b. Are not achievable given they lie below the efficient frontier
    c. Have lower expected returns than achievable portfolios with the same risk
4. The highest Sharpe ratio may be achieved
    a. Anywhere along the CAL tangent to the IOS
    b. Only at the CAL's point of tangency with the efficient frontier
    c. On the CAL above the point of tangency with the efficient frontier

NOTES & REFERENCES

[1] The formulas for the weights are obtained by solving the optimization problem

$$Max_{w_i}\, S_p = \frac{E(r_p) - r_f}{\sigma_p},$$

subject to the constraint that the weights w sum to 1. Plug in $w_1 E(r_1) + w_2 E(r_2)$ for the expected return of the portfolio $E(r_p)$, then differentiate with respect to w_2. Set equal to zero, and solve for w_1.

[2] CFA Question answers: 1)A, 2)C , 3)C , 4)A